STUDENT SUCCESS CENTER
THE UNIVERSITY OF ALABAMA IN HUNTSVILLE

CAL C: LINE INTEGRALS & VECTOR FIELDS

Line Integrals and Vector Fields:

- Line integrals are used as a general form of integration over a curve C rather than an interval. The curve needs to be parameterized using a ray that traces location as a function of t:

$$\boldsymbol{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k} \quad \text{for } a \le t \le b \quad \text{such that } f(x,y,z) = f\big(g(t), h(t), k(t)\big)$$

- Knowing that $\left|\frac{ds}{dt}\right| = |\boldsymbol{v}(t)|$, a line integral is written:

$$\int_C f(x,y,z)ds = \int_a^b f\big(g(t), h(t), k(t)\big) \cdot |\boldsymbol{v}(t)|\, dt$$

- Line integrals can be used in vector fields to find work, flux, and more. A vector field is defined:

$$F(x,y,z) = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\boldsymbol{k}$$

- One notable vector field is the gradient field, defined by the gradient vector F of a scalar function f**:**

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

- Knowing that the tangent vector is defined as $T = \frac{d\boldsymbol{r}}{ds}$, which defines the forward motion of the path, the line integral of a vector field $\boldsymbol{F}$ over path $\boldsymbol{r}(t)$ can be written:

$$\int_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds = \int_C \left(\boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{ds}\right) ds = \int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}\big(\boldsymbol{r}(t)\big) \cdot \frac{d\boldsymbol{r}}{dt} dt$$

- It is known that this integral is equal to the work done by a force $\boldsymbol{F}$ over a curve C from a to b as well as the flow of a fluid along the curve C.

- The flux of $\boldsymbol{F}$ across the curve C is defined by the scalar component of the fluid's velocity in the direction of the curve's *outward facing normal vector* (while the tangential vector leads to flow **along** the curve, flux is concerned with flow **across** the curve).

$$Flux = \int_C \boldsymbol{F} \cdot \boldsymbol{n}\, ds = \int_C \boldsymbol{F} \cdot (\boldsymbol{T} \times \mathbf{k})\, ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds = \oint_C M\, dy - N\, dx$$

For more information, visit a tutor. All appointments are available in-person at the Student Success Center, located in the Library, or online. Adapted from Hass, J., Weir, M.D., & Thomas, G.B. (2012). *University Calculus: Early Transcendentals* (2nd ed.). Boston: Pearson Education.